A new type of intermittent transition to chaos

1983 J. Phys. A: Math. Gen. 16 L109

(http://iopscience.iop.org/0305-4470/16/4/002)

J. Phys. A: Math. Gen. 16 (1983) L109–L112. Printed in Great Britain

LETTER TO THE EDITOR

A new type of intermittent transition to chaos

A S Pikovsky
Institute of Applied Physics, Academy of Sciences of the USSR, Gorky, USSR

Received 8 December 1982

Abstract. An intermittent transition to chaos in the presence of symmetry is investigated. Statistical properties of intermittency are found, including the case of external noise.

There has recently been a remarkable interest in the transition to chaos in dissipative dynamical systems. The most commonly discussed routes are a period-doubling (Feigenbaum) transition (see the review by Eckmann (1981)) and the transition via intermittency first described by Manneville and Pomeau (1980). Intermittency is in fact a regime with long-lived nearly periodic laminar phases interrupted by turbulent bursts. This regime results from the collision of stable and unstable periodic cycles. Statistical properties of Pomeau–Manneville's intermittency were described by Manneville (1980), Eckmann *et al* (1981) and Hirsch *et al* (1982a, b).

In this letter we describe a new type of intermittent route to chaos. This bifurcation occurs when two unstable cycles merge with a stable one to produce an unstable limit cycle. Such a transition can be observed in systems with a symmetry (e.g. in hydrodynamic systems with symmetrical boundary conditions), if stable and unstable cycles have different structures of symmetry.

The starting point is a one-dimensional mapping of the interval $-1 \leqslant z \leqslant 1$ into itself (figure 1)

$$z_{i+1} = F(z_i) = \begin{cases} f(z_i) = z_i^{0.1} + bz_i - 1, & z_i > 0, \\ -f(-z_i), & z_i < 0. \end{cases} \tag{1}$$

Such a symmetrical mapping describes, for example, the dynamics of the well known Lorenz system (see Williams 1979).

For $b < b_c = 0.382\ldots$ the mapping (1) has a stable period-2 cycle $(z^0, -z^0)$, where $z^0 = F^2(z^0)$. In a supercritical region $b > b_c$ this cycle is unstable, and successive interactions of (1) have the form of laminar phases separated by turbulent bursts (figure 2). The statistical characteristic of most interest is the distribution of durations of laminar phases, which may be obtained by considering only the system dynamics in the nearest vicinity of the cycle $(z^0, -z^0)$. A small displacement $y = z - z^0$ obeys the mapping

$$y_{i+2} = (1+\varepsilon)y_i + ay_i^3 + g\xi_i. \tag{2}$$

Here

$$\varepsilon = \mathrm{d}F^2(z^0)/\mathrm{d}z - 1 = \text{constant}\,(b - b_c)$$

is the bifurcational parameter,

$$a = \tfrac{1}{6}\,\mathrm{d}^3F^2(z^0)/\mathrm{d}z^3$$

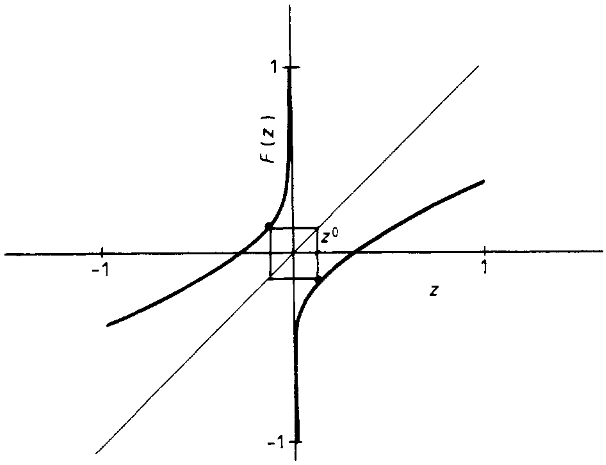

Figure 1. The symmetrical mapping (1) at $b = b_c$.

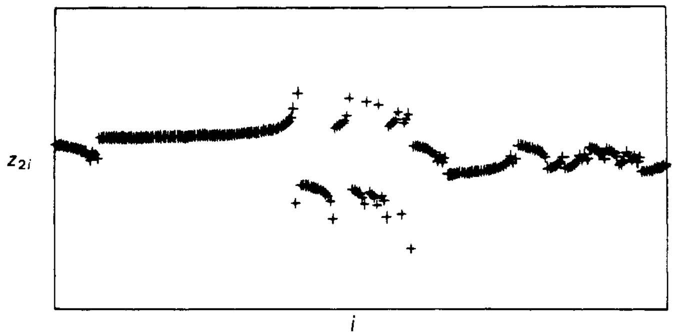

Figure 2. Intermittent regime, obtained in numerical simulation of (1) for $b = b_c + 0.01$; only even points are presented.

and a term y_i^2 vanishes due to symmetry. In the mapping (2) external noise is taken into account by adding $g\xi_i$ to the right-hand side, where ξ_i is the sequence of random variables with $\langle\xi\rangle = 1 - \langle\xi^2\rangle = 0$. Setting $x = a^{1/2}y$, we finally obtain

$$x_{n+1} = x_n + \varepsilon x_n + x_n^3 + \sigma\xi_n \tag{3}$$

where $n = 2i$, $\sigma = ga^{1/2}$. Equation (3) is the basic equation for investigation of the intermittency (see figure 3).

Dynamical case ($\sigma = 0$). If $\varepsilon \ll 1$ we may approximate (3) by a differential equation

$$\mathrm{d}x/\mathrm{d}t = \varepsilon x + x^3. \tag{4}$$

This is easily integrated to yield

$$t(x_0) = (2\varepsilon)^{-1} \ln[(x_0^2 + \varepsilon)/x_0^2] \tag{5}$$

for the time of motion from an initial point x_0 to infinity. Using the uniform distribution of initial points in the interval $-1 < x_0 < 1$, we obtain from (5) the distribution of

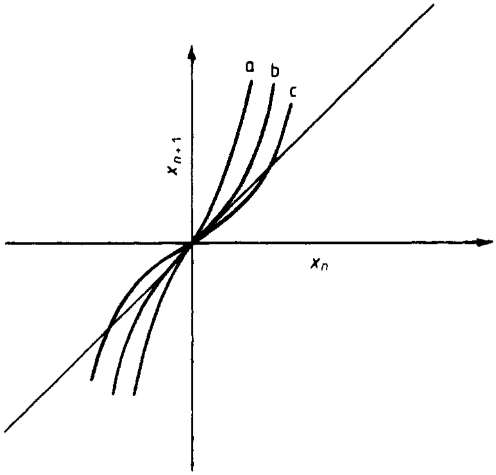

Figure 3. Evolution of the mapping (3) at the transition to chaos. a, $\varepsilon > 0$; b, $\varepsilon = 0$; c, $\varepsilon < 0$.

durations of laminar phases

$$W(t) = \begin{cases} \varepsilon^{3/2} \exp(2\varepsilon t)(\exp(2\varepsilon t) - 1)^{-3/2}, & \frac{1}{2} < t < \infty, \\ 0, & t < \frac{1}{2}. \end{cases} \tag{6}$$

The average time of the laminar phase $T(\varepsilon)$ and dispersion $D(\varepsilon)$ are then readily obtained:

$$T(\varepsilon) = \int_{1/2}^{\infty} tW(t)\,\mathrm{d}t \simeq \tfrac{1}{2}\pi\varepsilon^{-1/2}, \tag{7}$$

$$D(\varepsilon) = \int_{1/2}^{\infty} (t - T(\varepsilon))^2 W(t)\,\mathrm{d}t \simeq \pi(\ln 2)\varepsilon^{-3/2}. \tag{8}$$

Note that (7) and (8) yield

$$D(\varepsilon)/T^2(\varepsilon) \simeq [4(\ln 2)/\pi]\varepsilon^{-1/2}, \tag{9}$$

i.e. near the bifurcation point, fluctuations of durations of laminar phases are very large, unlike in the Pomeau–Manneville case, where fluctuations do not grow as $\varepsilon \to 0$.

The influence of external noise. If $\sigma \neq 0$, but $|\varepsilon|, \sigma \ll 1$, we may derive from the stochastic mapping (3), following Hirsch *et al* (1982a), the Fokker–Planck equation

$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial x}[(\varepsilon x + x^3)w] + \frac{\sigma^2}{2}\frac{\partial^2 w}{\partial x^2}. \tag{10}$$

According to Stratonovich (1963), for a Markov process (10) averaged over ξ, the duration of the laminar phase initiated at x_0 is

$$\tau(x_0) = -\frac{2}{\sigma^2}\int_{-\infty}^{x_0} \mathrm{d}p \int_0^p \mathrm{d}q\, \exp\{\sigma^{-2}[\varepsilon(q^2 - p^2) + \tfrac{1}{2}(q^4 - p^4)]\}. \tag{11}$$

Using again the uniform distribution of initial points, we finally obtain

$$T(\varepsilon, \sigma) = \tfrac{1}{2}\int_{-1}^{1} \tau(x_0)\,\mathrm{d}x_0 = \sigma^{-1/2}G(\rho), \qquad \rho = \varepsilon\sigma^{-1}, \tag{12}$$

where G is the universal function

$$G(\rho)=2^{-1/2}\int_0^1 (1-v)^{-1/2}\,\mathrm{d}v\int_0^\infty u^2\exp[-v(\rho u^2+\tfrac{1}{2}u^4)]\,\mathrm{d}u$$

$$=\pi^{1/2}2^{-7/4}\sum_{k=0}^{\infty}(-\rho)^k 2^{k/2}\Gamma\left(\frac{2k+1}{4}\right)(k!)^{-1}. \qquad (13)$$

The power expansion series (13) converges for all ρ, but for $|\rho|\gg 1$ the following asymptotic expressions are more useful:

$$G(\rho)\simeq\tfrac{1}{2}\pi\rho^{-1/2} \qquad \text{for } \rho\gg 1, \qquad (14)$$

$$G(\rho)\simeq\pi 2^{-1/2}|\rho|^{-1/2}\exp(\tfrac{1}{2}|\rho|^2) \qquad \text{for } \rho\ll -1. \qquad (15)$$

Note that (14) is consistent with (7):

$$\lim_{\sigma\to 0} T(\varepsilon,\sigma)=\sigma^{-1/2}\lim_{\rho\to\infty} G(\rho)=\tfrac{1}{2}\pi\varepsilon^{-1/2}=T(\varepsilon).$$

It follows from (15) that for the subcritical region $\rho\ll -1$ the probability of the appearance of a turbulent burst is exponentially small.

In conclusion we would like to point out that powers in (7), (12) may be obtained using a renormalisation group approach, developed earlier for Pomeau–Manneville intermittency by Hirsch *et al* (1982b) and Hu and Rudnick (1982).

References

Eckmann J P 1981 *Rev. Mod. Phys.* **53** 643

Eckmann J P, Thomas L and Wittwer P 1981 *J. Phys. A: Math. Gen.* **14** 3153

Hirsch J E, Huberman B A and Scalapino D J 1982a *Phys. Rev.* A **25** 519

Hirsch J E, Nauenberg M and Scalapino D J 1982b *Phys. Lett.* **87A** 391

Hu B and Rudnick J 1982 *Phys. Rev. Lett.* **48** 1645

Manneville P 1980 *J. Physique* **41** 1235

Manneville P and Pomeau Y 1980 *Physica* D **1** 219

Pikovsky A S 1981 *Preprint*

Stratonovich R L 1963 *Topics in the theory of random noise* (New York: Gordon and Breach)

Williams R F 1979 in *Bifurcation theory and applications in scientific disciplines* ed O Gurel and O E Rössler *Ann. NY Academy Sciences* **316** 393